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PROPERTIES OF CERTAIN INTEGRAL OPERATOR

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Abstract

Let $A(p)$ denote the class of functions $f(z)$ which are analytic and p–valent in the unit disk U. A new subclass $\Omega(\alpha,\beta;\gamma)$ of $A(p)$ consisting of analytic and p–valent functions $f(z)$ associated with the certain integral operator Q_β^α which is the generalization of the integral operator studied by I.B.Jung, Y.C.Kim and H.M.Srivastava (J. Math. Anal. Appl. **248**(2000), 475 - 481) is introduced. Some interesting properties of the operator Q_β^α for functions $f(z)$ belonging to $A(p)$ are investigated.

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1. Introduction.

Let $A(p)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \qquad (p \in N = \{1, 2, 3, \cdots\}) \tag{1.1}$$

which are analytic and p-valent in the unit disk $U = \{z : z \in C \text{ and } |z| < 1\}$. Let $S_p^*(\gamma)$ denote the class of functions $f(z)$ of the form (1.1) which satisfy the condition

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > p\gamma$$

for $0 \leq \gamma < 1$ and $z \in U$. A function in $S_p^*(\gamma)$ is called p-valent starlike of order γ in U.

Let $f(z)$ and $g(z)$ be analytic in U. Then we say that the function $g(z)$ is subordinate to $f(z)$ if there exists an analytic function $w(z)$ in U such that $|w(z)| < 1 (z \in U)$ and $g(z) = f(w(z))$. For this relation the symbol $g(z) \prec f(z)$ is used. In case $f(z)$ is univalent in U we have that the subordination $g(z) \prec f(z)$ is equivalent to $g(0) = f(0)$ and $g(U) \subset f(U)$.

Recently, Jung, Kim and Srivastava [3] introduced the following integral operator:

$$Q_\beta^\alpha f(z) = \binom{\alpha+\beta}{\beta} \frac{\alpha}{z^\beta} \int_0^z (1 - \frac{t}{z})^{\alpha-1} t^{\beta-1} f(t)dt$$

$$(\alpha > 0, \beta > -1; f \in A(1)). \tag{1.2}$$

They also showed that

$$Q_\beta^\alpha f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(\beta+n)\Gamma(\alpha+\beta+1)}{\Gamma(\beta+\alpha+n)\Gamma(\beta+1)} a_n z^n.$$

It follows from (1.3) that one can define the operator Q_β^α for $\alpha \geq 0$ and $\beta > -1$. Some interesting subclasses of analytic function, associated with the operator Q_β^α, have been considered recently by Jung et al.[3], Aouf et al.[1], Li[5], Liu[6] and others.

Motivated by Jung, Kim and Srivastava's work [3], we now consider a linear operator $Q_\beta^\alpha : A(p) \to A(p)$ as following:

$$Q_\beta^\alpha f(z) = \binom{p+\alpha+\beta-1}{p+\beta-1} \frac{\alpha}{z^\beta} \int_0^z \left(1 - \frac{t}{z}\right)^{\alpha-1} t^{\beta-1} f(t)dt$$

$$(\alpha \geqslant 0, \beta > -1; f \in A(p)). \tag{1.3}$$

We note that

$$Q_\beta^\alpha f(z) = z^p + \sum_{n=1}^{\infty} \frac{\Gamma(p+n+\beta)\Gamma(p+\alpha+\beta)}{\Gamma(p+n+\alpha+\beta)\Gamma(p+\beta)} a_{p+n} z^{p+n}$$

$$(\alpha \geq 0, \beta > -1; f \in A(p)). \tag{1.4}$$

It is easily verified from the definition (1.4) that

$$z(Q_\beta^\alpha f(z))' = (\alpha+\beta+p-1)Q_\beta^{\alpha-1}f(z) - (\alpha+\beta-1)Q_\beta^\alpha f(z). \tag{1.5}$$

When $p = 1$, the identity (1.5) is given in [3]. One can easily see that the operator Q_β^α has an inverse operator $Q_{\beta+\alpha}^{-\alpha}$ and Q_β^0 is an unit operator.

A function $f(z) \in A(p)$ is said to be in the class $\Omega(\alpha, \beta; \gamma)$ if it satisfies the condition

$$\frac{z(Q_\beta^\alpha f(z))'}{Q_\beta^\alpha f(z)} + \frac{pz^p}{1-z^p} \prec \frac{p+p(1-2\gamma)z}{1-z} \tag{1.6}$$

for all $z \in U$ and $0 \leq \gamma < 1$.

In this paper, we shall show the extreme points of the closed convex hull of the class $\Omega(\alpha, \beta; \gamma)$. It is then used to determine the coefficient bounds.

In the sequel, we denote the closed convex hull of a class H by coH. Also, let $E(coH)$ denote the set of all extreme points of H.

2. Main Results.

In order to derive our main results, we shall need the following lemmas.

Lemma 1 ([4]). $E(coS_p^*(\alpha))$ consists of the functions given by

$$\frac{z^p}{(1-xz)^{2p(1-\gamma)}} = z^p + \sum_{n=1}^{\infty} \frac{(2p-2p\gamma)_n}{n!} x^n z^{p+n} \quad (z \in U), \tag{2.1}$$

where $(a)_n = a(a+1)\cdots(a+n-1), x \in C$ and $|x| = 1$.

Lemma 2 ([9]). The function $(1-z)^\rho \equiv e^{\rho \log(1-z)}, \rho \neq 0$, is univalent in U if and only if ρ is either in the closed disk $|\rho - 1| \leq 1$ or in the closed disk $|\rho+1| \leq 1$.

Lemma 3 ([7]). Let $q(z)$ be univalent in U and let $\theta(w)$ and $\phi(w)$ be analytic in a domain D containing $q(U)$ with $\phi(w) \neq 0$ when $w \in q(U)$. Set $Q(z) = zq'(z)\phi(q(z)), h(z) = \theta(q(z)) + Q(z)$ and suppose that

(1)$Q(z)$ is starlike (univalent) in U;

(2) $Re\left\{\frac{zh'(z)}{Q(z)}\right\} = Re\left\{\frac{\theta'(q(z))}{\phi(q(z))} + \frac{zQ'(z)}{Q(z)}\right\} > 0 \quad (z \in U).$

If $p(z)$ is analytic in U, with $p(0) = q(0), p(U) \subset D$ and

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)) = h(z), \tag{2.2}$$

then $p(z) \prec q(z)$ and $q(z)$ is the best dominant.

Theorem 1. A function $f(z) \in A(p)$ is in $\Omega(\alpha,\beta;\gamma)$ if and only if $f(z)$ can be expressed as

$$f(z) = Q^{-\alpha}_{\beta+\alpha}\left\{z^p(1-z^p)exp[-2p(1-\gamma)\int_X \log(1-xz)d\mu(x)]\right\}, \tag{2.3}$$

where μ is a probability measure defined on the unit circle $X = \{x : |x| = 1\}$.

Proof. Let $f(z) \in \Omega(\alpha,\beta;\gamma)$. Then by Herglotz formula [2], we have

$$\frac{z(Q^\alpha_\beta f(z))'}{Q^\alpha_\beta f(z)} + \frac{pz^p}{1-z^p} = p(1-\gamma)\int_X \frac{1+xz}{1-xz}d\mu(x) + p\gamma, \tag{2.4}$$

where μ is a probability measure defined on the unit circle $X = \{x : |x| = 1\}$. By means of the identity

$$\frac{d}{dz}\log\frac{Q^\alpha_\beta f(z)}{z^p(1-z^p)} = \frac{1}{z}\left[\frac{z(Q^\alpha_\beta f(z))'}{Q^\alpha_\beta f(z)} + \frac{pz^p}{1-z^p} - p\right], \tag{2.5}$$

(2.4) yields

$$Q^\alpha_\beta f(z) = z^p(1-z^p)\exp[-2p(1-\gamma)\int_X \log(1-xz)d\mu(x)]. \tag{2.6}$$

Thus

$$f(z) = Q^{-\alpha}_{\beta+\alpha}\{z^p(1-z^p)\exp[-2p(1-\gamma)\int_X \log(1-xz)d\mu(x)]\}.$$

Now the proof is complete.

Theorem 2. Let $0 \le \gamma_1 < \gamma_2 < 1$, then $\Omega(\alpha,\beta;\gamma_2) \subset \Omega(\alpha,\beta;\gamma_1)$.

Proof. We define a linear operator on $\Omega(\alpha,\beta;\gamma)$ as following:

$$T_\gamma(f) = \frac{Q^\alpha_\beta f(z)}{1-z^p} \quad (z \in U). \tag{2.7}$$

Then T_γ is a linear homeomorphism from $\Omega(\alpha,\beta;\gamma)$ to $S^*_p(\gamma)$. It is well-known that $S^*_p(\gamma_2) \subset S^*_p(\gamma_1)$ for $0 \le \gamma_1 < \gamma_2 < 1$. The result follows immediately.

Theorem 3. (i) The extreme points of $co\Omega(\alpha,\beta;\gamma)$ are given by the functions

$$f_x(z) = Q^{-\alpha}_{\beta+\alpha}\left\{\frac{z^p(1-z^p)}{(1-xz)^{2p(1-\gamma)}}\right\}$$

$$(x \in C, |x| = 1; z \in U). \tag{2.8}$$

$$(ii)\ \ Co\ \Omega(\alpha,\beta;\gamma)=\{f : f(z)=\int_X f_x(z)d\mu(x)\}, \tag{2.9}$$

where μ varies over the probability measures defined on the unit circle X.

Proof. Since T_γ defined by (2.7) is a linear homeomorphism from $\Omega(\alpha,\beta;\gamma)$ to $S_p^*(\gamma)$, it preserves extreme points. By making use of Lemma 1, the results follow at once.

According to Theorem 3 and Lemma 1, we have the following corollaries.

Corollary 1. Let $f(z)=z^p+\sum\limits_{n=1}^{\infty}a_{p+n}z^{p+n}\in\Omega(\alpha,\beta;\gamma)$. Then

$$|a_{p+n}|\le\begin{cases}\frac{(2p-2p\gamma)_n}{n!}\cdot\frac{\Gamma(p+n+\alpha+\beta)\Gamma(p+\beta)}{\Gamma(p+n+\beta)\Gamma(p+\alpha+\beta)}, & 1\le n<p,\\ \frac{(2p-2p\gamma)_{n-p}\left|\prod\limits_{k=1}^{p}(2p-2p\gamma+n-k)-\prod\limits_{k=1}^{p}(n-p+k)\right|}{n!}\cdot\frac{\Gamma(p+n+\alpha+\beta)\Gamma(p+\beta)}{\Gamma(p+n+\beta)\Gamma(p+\alpha+\beta)}, & n\ge p.\end{cases}$$

The result is sharp.

Corollary 2. Let $f(z)=z^p+\sum\limits_{n=1}^{\infty}a_{p+n}z^{p+n}\in\Omega(\alpha,\beta;\gamma)$. Then for $|z|=r<1$.

$$|f(z)|\le r^p+\sum_{n=1}^{p-1}\frac{(2p-2p\gamma)_n}{n!}\cdot\frac{\Gamma(p+n+\alpha+\beta)\Gamma(p+\beta)}{\Gamma(p+n+\beta)\Gamma(p+\alpha+\beta)}r^{p+n}$$

$$+\sum_{n=p}^{\infty}\frac{(2p-2p\gamma)_{n-p}\left|\prod\limits_{k=1}^{p}(2p-2p\gamma+n-k)-\prod\limits_{k=1}^{p}(n-p+k)\right|}{n!}\cdot\frac{\Gamma(p+n+\alpha+\beta)\Gamma(p+\beta)}{\Gamma(p+n+\beta)\Gamma(p+\alpha+\beta)}r^{p+n}.$$

The result is sharp.

Theorem 4. Let $f(z)\in\Omega(\alpha,\beta;\gamma)$. Let ρ be a complex number with $\rho\ne 0$ and satisfy either $|2p\rho(1-\gamma)+1|\le 1$ or $|2p\rho(1-\gamma)-1|\le 1$. Then

$$\left(\frac{Q_\beta^\alpha f(z)}{z^p(1-z^p)}\right)^\rho\prec\frac{1}{(1-z)^{2p\rho(1-\gamma)}}=q(z)\quad(z\in U), \tag{2.10}$$

where $q(z)$ is the best dominant.

Proof. Let

$$p(z)=\left(\frac{Q_\beta^\alpha f(z)}{z^p(1-z^p)}\right)^\rho, \tag{2.11}$$

then $p(z)$ in analytic is U with $p(0)=1$. Differentiating (2.11) logarithmically we have

$$\frac{zp'(z)}{p(z)}=\rho\left(\frac{z(Q_\beta^\alpha f(z))'}{Q_\beta^\alpha f(z)}+\frac{pz^p}{1-z^p}-p\right). \tag{2.12}$$

Since $f(z) \in \Omega(\alpha, \beta; \gamma)$, (2.12) is equivalent to

$$p + \frac{zp'(z)}{\rho p(z)} \prec \frac{p + p(1-2\gamma)z}{1-z} = h(z). \tag{2.13}$$

If we take

$$q(z) = \frac{1}{(1-z)^{2p\rho(1-\gamma)}}, \theta(w) = p \text{ and } \phi(w) = \frac{1}{\rho w}, \tag{2.14}$$

then $q(z)$ is univalent by the condition of the theorem and Lemma 2. It is easy to show that $q(z), \theta(w)$ and $\phi(w)$ satisfy the conditions of Lemma 3. Since

$$Q(z) = zq'(z)\phi(q(z)) = \frac{2p(1-\gamma)z}{1-z} \tag{2.15}$$

is univalent starlike in U and

$$h(z) = \theta(q(z)) + Q(z) = \frac{p + p(1-2\gamma)z}{1-z}, \tag{2.16}$$

it may be readily checked that the conditions (1) and (2) of Lemma 3 are satisfied. Thus the result follows from (2.13) immediately.

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